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Dynamic Programming with Homogeneous Functions

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We show that the basic existence, uniqueness, and convergence results of dynamic programming hold when the return function is homogeneous of degree $\theta \leqslant 1$ and the constraints are homogeneous of degree one. *Journal of Economic Literature Classification* Number: C61. © 1998 Academic Press

A major limitation in applying the tools of dynamic programming to many economic problems has been the lack of a general theory for the case where returns are unbounded. Except for a few special cases where closed form solutions are available (linear-quadratic models and the log-Cobb–Douglas growth model being the leading examples), recursive methods have not been useful. Here we establish that the basic existence, uniqueness, and convergence results of dynamic programming, which are fundamental for both theoretical and computational purposes, hold for a useful family of unbounded problems, those that are homogeneous.

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Many problems in economics are conveniently modeled with return functions that are homogeneous of degree $\theta \leqslant 1$ and constraints that are homogeneous of degree one. For example, in much of the endogenous growth literature (Lucas [8], Jones and Manuelli [7], Rebelo [11], and many others), a utility function with a constant elasticity of intertemporal substitution is used,

$$U(c) = \frac{c^{\theta}}{\theta}, \qquad \theta < 1,$$

together with technologies that display constant returns to scale. (The case $\theta = 0$ is interpreted as $U(c) = \ln(c)$.) Preferences in this class are used because homogeneous functions are the only ones consistent with balanced growth. The fertility models studied in Barro and Becker [2], Mulligan [9], and Alvarez [1] also have this form, as do models in many other areas of economics.

This paper shows that the basic results of dynamic programming hold for problems of this sort. Two types of results are established. First we show that the Principle of Optimality applies—that solutions to the dynamic program coincide exactly with solutions to the original problem. We then show that the Bellman equation has a solution, that this solution is unique, and that the operator defined by the Bellman equation provides an algorithm for finding it. To establish these results, different assumptions and different lines of proof are needed for the cases $0 < \theta \leqslant 1$, $\theta < 0$, and $\theta = 0$.

In each case one critical step is finding restrictions that bound the growth rate of the state variables from above or below along some or all feasible paths, to insure that total returns along the optimal path(s) are bounded. These restrictions are different for each case. With this done the arguments for the case $0 < \theta \leqslant 1$ parallel very closely those for the case where returns are bounded. For the cases $\theta < 0$ and $\theta = 0$, however, some new complications arise. First, in these cases it is natural to asssume that returns diverge to minus infinity along some feasible paths. Hence the return function defined over the set of feasible sequences is not continuous, and Berge's theorem cannot be applied. In addition, since the value function diverges to minus infinity at the origin, Berge's theorem cannot be used to establish that the Bellman operator preserves continuity. Finally, for the case $\theta < 0$ the contraction mapping theorem may not apply. Consequently we will need additional restrictions on the constraints and return function in the cases $\theta < 0$ and $\theta = 0$.

The rest of the paper is organized as follows. In Section 1 we define an appropriate normed space of functions, and in Sections 2–4 we deal in turn with the cases $0 < \theta \leqslant 1$, $\theta < 0$, and $\theta = 0$. Various extensions are discussed briefly in Section 5.

1. PRELIMINARIES

Consider the problem

$$v^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \qquad t = 0, 1, 2, ...,$$

$$x_0 \in X \text{ given},$$

and the corresponding Bellman equation

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)], \quad \text{all } x \in X, \tag{1}$$

where Γ is homogeneous of degree one and F is homogeneous of degree $\theta \leqslant 1$. Define the product space $Z = X \times X \times X \times \cdots$, and define Π: $X \to Z$ by

$$\Pi(x_0) = \{\underline{x} = (x_0, x_1, ...): x_{t+1} \in \Gamma(x_t), t = 0, 1, ...\}, \quad \text{all } x_0 \in X,$$

so $\Pi(x_0)$ is the set of all sequences in Z that are feasible from x_0. For any $\underline{x} \in \Pi(x_0)$, let $u(\underline{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ be the total discounted returns. We can then write the problem above more compactly as

$$v^*(x_0) = \sup_{\underline{x} \in \Pi(x_0)} u(\underline{x}). \tag{2}$$

We will later add restrictions, different for each range for θ, that insure u is well defined. Once such restrictions are added, since Γ and F are homogeneous, so are Π and u, and hence so is v^*: $v^*(\lambda x_0) = \lambda^\theta v^*(x_0)$.

The following space of functions is useful for analyzing (1) if $\theta \neq 0$. Let $X \subseteq \mathbf{R}^\ell$ be a cone, and let $H(X, \theta)$ be the linear space of functions f: $X \to \mathbf{R}$ that are homogeneous of degree θ, bounded in the norm

$$\|f\| = \sup_{\|x\| = 1,\, x \in X} |f(x)|, \tag{3}$$

and continuous, except at the origin if $\theta < 0$. With this norm $H(X, \theta)$ is a complete metric space. Note that there is a one-to-one relationship between elements of $H(X, \theta)$ and elements of the set of bounded, continuous functions defined on the intersection of X with the unit circle. If $\theta < 0$, then functions in $H(X, \theta)$ diverge as $\|x\| \to 0$.

Let $H^+(X, \theta)$ and $H^-(X, \theta)$ be the closed subsets of $H(X, \theta)$ containing the nonnegative and nonpositive functions, respectively. For $\theta > 0$ we will assume that returns are positive and use the space of functions $H^+(X, \theta)$; for $\theta < 0$ we will assume that returns are negative and use $H^-(X, \theta)$.

For any $f \in H$ and $a \in \mathbf{R}$, let $f + a$ denote the function

$$(f+a)(x) = f(x) + a\,\|x\|^\theta,$$

so $f + a$ is also continuous and homogeneous of degree θ. Also note that homogeneity implies

$$f(x) = \|x\|^\theta f\left(\frac{x}{\|x\|}\right) \leqslant \|x\|^\theta\,\|f\|. \tag{4}$$

2. HOMOGENEOUS FUNCTIONS: $0 < \theta \leqslant 1$

For the case $0 < \theta < 1$ the arguments are analogous to those for the case $\theta = 1$ in Stokey, Lucas, and Prescott [12, Ch. 4.3], so they are sketched briefly or omitted. We will use the following restrictions on the state space, feasibility constraints, and return function. The main issues are to bound the growth rate of the state variable and to bound the return function.

Assumption 1. a. $0 < \theta \leqslant 1$ and $X \subseteq \mathbf{R}^\ell$ is a cone;

b. the correspondence $\Gamma\colon X \to X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone: i.e., $\Gamma(0) = \{0\}$, and

$$y \in \Gamma(x) \Rightarrow \lambda y \in \Gamma(\lambda x), \quad \text{all } \lambda > 0, \quad \text{all } x \in X;$$

c. $\beta > 0$, and there exists $\alpha > 0$ with $\gamma \equiv \alpha^\theta \beta < 1$, such that

$$\|y\| \leqslant \alpha \|x\|, \quad \text{all } (x, y) \in A;$$

d. $F\colon A \to \mathbf{R}_+$ is continuous and homogeneous of degree θ, and for some $0 < B < \infty$,

$$F(x, y) \leqslant B(\|x\| + \|y\|)^\theta, \quad \text{all } (x, y) \in A.$$

The results of this section also hold for $\theta > 1$, but it is difficult to think of economic problems where that assumption is useful. Assumption 1c, which is the Brock–Gale [5] condition for the existence of optimal paths, bounds the rate of growth of feasible sequences by $\beta^{-1/\theta}$. Notice that the discount factor β may be greater than one, provided that $\alpha < 1$ is small enough so that $\gamma = \beta\alpha^\theta$ is less than one. The condition $\alpha < 1$ implies that the state variable *shrinks* over time along every feasible path. It is difficult to think of an example where this combination of assumptions makes sense, but the mathematics permits it. Assumption 1d is equivalent to assuming that

$F(x, y)$ is bounded for $\|x\| = 1$ and $y \in \Gamma(x)$, and it implies that $F(0, 0) = 0$. If X is closed, then A is also closed, and the continuity of F implies the existence of some $B < \infty$ satisfying the required inequality. Notice that under Assumption 1 u is well defined (the sum converges).

Theorem 1 establishes the relationship between solutions to (1) and (2), and Theorem 2 establishes existence, uniqueness and convergence results for (1).

THEOREM 1. *Let* $(\theta, X, \Gamma, \beta, F)$ *satisfy Assumption* 1. *Then*

a. $v^* \in H^+(X, \theta)$;

b. v^* satisfies (1);

c. $\underline{x}^* \in \Pi(x_0)$ attains the supremum in (2) for initial state x_0 if and only if

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \qquad t = 0, 1, 2, ... \tag{5}$$

Proof. We have already noted that v^* is homogeneous, clearly it is nonnegative, and Assumptions 1c and 1d imply that it is bounded in the norm in (3). To establish continuity, the argument in [7, Proposition 1] applies. First note that Π: $X \to Z$ is compact valued and continuous in the product topology. In addition, using the bound on the growth rate of $\|x_t\|$ in Assumption 1c, it follows from the Lebesgue Dominated Convergence Theorem that u: $Z \to \mathbf{R}_+$ is continuous in that topology. Hence by the Berge's Theorem of the Maximum, v^* is continuous, and (a) holds.

Assumption 1c and (4) together imply that

$$\lim_{t \to \infty} \beta^t |v^*(x_t)| \leqslant \lim_{t \to \infty} \beta^t \|x_t\|^\theta \|v^*\| \leqslant \lim_{t \to \infty} \gamma^t \|x_0\|^\theta \|v^*\| = 0,$$

$$\text{all } \underline{x} \in \Pi(x_0), \quad \text{all } x_0 \in X,$$

so claims (b) and (c) follow from Theorems 4.2, 4.4, and 4.5 in [12]. ∎

The proof of Theorem 2 uses Lemma 1, which is a corollary to Boyd's [4] Weighted Contraction Theorem (with $\phi(x) = \|x\|^\theta$). In some applications it is useful to apply the result to closed subspaces of $H^+(X, \theta)$, for example the subspace containing weakly quasiconcave functions.

LEMMA 1 (Boyd). *Let* $\theta \in \mathbf{R}$, *with* $\theta \neq 0$; *let* $X \subseteq \mathbf{R}^\ell$ *be a cone, excluding the origin if* $\theta < 0$; *and let* $J(X, \theta)$ *be a space of functions that are homogeneous of degree* θ *and bounded in the norm in* (3). *Let* T: $J(X, \theta) \to J(X, \theta)$ *be an operator satisfying*

a. (*monotonicity*) *$f, g \in J(X, \theta)$ and $f \leqslant g$, implies $Tf \leqslant Tg$;*

b. (*discounting*) *there exists $\gamma \in (0, 1)$ such that*

$$T(f+a) \leqslant (Tf) + \gamma a, \quad \text{all } f \in J(X, \theta), \quad a \geqslant 0.$$

Then T is a contraction of modulus γ.

THEOREM 2. *Let $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption* 1, *and define the operator T on $H^+(X, \theta)$ by*

$$(Tf)(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta f(y)]. \tag{6}$$

Then

(a) *$T: H^+(X, \theta) \to H^+(X, \theta)$, T is a contraction of modulus γ, and v^* is the unique fixed point of T.*

(b) *The policy correspondence G defined by*

$$G(x) = \{ y^* \in \Gamma(x) \colon y^* \in \arg\max_{y \in \Gamma(x)} [F(x, y) + \beta v^*(y)]\}$$

is nonempty, compact-valued, u.h.c., and homogeneous of degree one.

Proof. Using Lemma 1, the proof is standard, see [12, Theorem 4.13].

The proof in Theorem 1 that v^* is continuous used Berge's theorem applied to the infinite horizon problem in (2). An alternate proof is also available. Let $B(X, \theta)$ be the linear space of functions $f: X \to \mathbf{R}$ that are homogeneous of degree θ and bounded in the norm in (3). Clearly v^* defined in (2) lies in $B(X, \theta)$ and clearly it is a fixed point of T. Then note that $B(X, \theta)$ is complete, T maps $B(X, \theta)$ into itself, and (by Lemma 1) T is a contraction. Hence v^* is the only fixed point of T in $B(X, \theta)$. Finally, applying Berge's theorem to the right side of (6), we find that T maps $H^+(X, \theta)$, a closed subspace of $B(X, \theta)$, into itself, so another application of Lemma 1 shows that the unique fixed point of T belongs to $H^+(X, \theta)$. Hence v^* is continuous.

Thus, in the case $\theta > 0$ the continuity of v^* can be established by either of two lines of argument: by showing that u is continuous and applying Berge's result to (2), or by establishing that T is a contraction and applying Berge's result to the right side of (6). For the case $\theta < 0$ neither of these arguments goes through: u is not continuous, so the first line fails, and T is not a contraction, so the second line also fails. Instead, an alternative argument, requiring an extra assumption, will be used in analyzing (2). For the logarithmic case $\theta = 0$, the first line fails—again because u is not continuous, but T is a contraction and the second line of proof can be used.

3. HOMOGENEOUS FUNCTIONS: $\theta < 0$

For $\theta < 0$ we will assume that returns are negative and use the space of functions $H^{-}(X, \theta)$. This case will require somewhat different assumptions on the feasible growth rate for the state. In addition, a number of other issues arise, which will require some minor additional assumptions.

If $\theta > 0$ the value of a homogeneous function $f(x_t)$ grows in absolute value as $\|x_t\|$ grows. Thus, to ensure that total discounted returns did not diverge along any feasible path, Assumption 1 put an upper bound on the growth rate of $\|x_t\|$ along every feasible path. If $\theta < 0$ the value of a homogeneous function grows in absolute value as $\|x_t\|$ *shrinks*. Thus, one possibility for this case would be to put a *lower* bound on the growth rate of $\|x_t\|$ along every feasible path. Most applications do not fit this restriction, however.

Instead, in most applications the return function takes nonpositive values, $F \leqslant 0$, so returns are bounded above by zero but are potentially unbounded below. Then, since total returns are being maximized, it is enough to assume that from every initial condition there is at least *one* feasible path along which returns do not diverge to minus infinity. Hence it suffices to assume that from every initial condition $x_0 \in X$, there is at least one feasible path $\underline{x} \in \Pi(x_0)$ along which $\|x_t\|$ does not shrink too quickly. This in turn implies that $\|x_t\|^{\theta}$ does not grow too quickly, so total discounted returns are bounded away from minus infinity.

The treatment of the origin is also a little delicate. Since the feasibility constraints are homogeneous of degree one, if the state reaches the origin it must remain there: $\Pi(0) = \{(0, 0, 0, ...)\}$. Moreover, since the return function is homogeneous of degree $\theta < 0$, $F(0, 0) = -\infty$. Hence the origin is a very undesirable point. Nevertheless, in many applications it is natural to include it as part of the feasible set, so we allow that possibility here.

Several entirely new complications arise when $\theta < 0$. Two involve Berge's theorem. For the case $\theta > 0$, the growth rate of the state was bounded above, so the Lebesgue Dominated Convergence Theorem provided a straightforward proof that u was continuous, and Berge's Theorem could be used to establish that v^* was continuous. For the present case we have no analogous bound on the growth rate of the state, so that line of argument cannot be used. Fatou's lemma provides one inequality, establishing that u is u.h.c., but that conclusion is not enough to let us apply Berge's theorem to (7).

In addition, when $\theta < 0$ the value function diverges to minus infinity at the origin. Therefore, if the origin is in the feasible set (i.e., if $0 \in \Gamma(x)$), the right side of (6) is not continuous in y, and Berge's theorem cannot be used to establish that T preserves continuity. Finally, when $\theta < 0$ the operator T need not be a contraction. We will present an example below that illustrates this possibility.

To deal with these various complications we will need several additional assumptions, all of them minor. One will be used to establish directly that v^* defined in (2) is continuous. The others will be used to show that T preserves continuity and that the fixed point of T is unique. Together, the continuity and monotonicity of T will allow us to establish a modified convergence result.

Assumption 2. a. $\theta < 0$ and $X \subseteq \mathbf{R}^{\ell}$ is a cone.

b. The correspondence $\Gamma: X \to X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone: i.e., $\Gamma(0) = \{0\}$, and

$$y \in \Gamma(x) \Rightarrow \lambda y \in \Gamma(\lambda x), \quad \text{all } \lambda > 0, \quad \text{all } x \in X.$$

c. $\beta > 0$, and there exists a continuous and homogeneous of degree one selection g from Γ and a number $\zeta > 0$ with $\gamma \equiv \beta \zeta^{\theta} < 1$ such that

$$\|g(x)\| \geqslant \zeta \, \|x\|, \quad \text{all } x \in X.$$

d. $F: A \backslash \{0\} \to \mathbf{R}_{-}$ is continuous and homogeneous of degree θ, and for some $0 < b < +\infty$,

$$F(x, y) \leqslant -b \, \|x\|^{\theta}, \quad \text{all } (x, y) \in A,$$

and $F(0, 0) = -\infty$.

e. If $y \in \Gamma(x)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - \hat{x}\| < \delta \qquad \text{implies} \quad (1 - \varepsilon) \, y \in \Gamma(\hat{x}).$$

Much of Assumption 2 is straightforward. As before the discount factor β may exceed one, provided that $\gamma = \beta \zeta^{\theta}$ is less than one. With $\theta < 0$, both can hold for $\zeta > 1$. That is, with $\theta < 0$, sustained growth is compatible with a discount factor that exceeds unity. The restriction $F < 0$ in part (2d) insures that $u(\underline{x})$ is well defined, although there may be many feasible paths along which returns diverge to $-\infty$. The growth restriction on the function g in part (2c) insures that for any initial state there exists at least one feasible path along which the growth rate of $\|x_{t+1}\| / \|x_t\|$ is bounded below by ζ. For this path total returns are bounded, and it follows immediately that v^* is bounded in the norm in (3). In anticipation of the boundedness argument it is convenient to define

$$-B = \inf_{x \in X, \, \|x\| = 1} \frac{F(x, g(x))}{\|x\|^{\theta} + \|g(x)\|^{\theta}}. \tag{7}$$

Then

$$F(x, g(x)) \geqslant -B(\|x\|^{\theta} + \|g(x)\|^{\theta}) \geqslant -B \, \|x\|^{\theta} \, (1 + \zeta^{\theta}), \quad \text{all } x \neq 0.$$

The assumption that F is continuous insures that B is finite. Clearly the homogeneity restriction on g is innocuous.

The more novel parts of Assumption 2 are the continuity restriction on the function g, the uniform bound away from zero in part (d), and the restriction in part (e). The continuity of g will be used to show that the operator T preserves continuity. The argument will use Lemma 2, which is an extension of Berge's Theorem to functions that take the value $-\infty$. This lemma requires the existence of such a function.

The restriction $-b<0$ in part (2d) ensures that $F(\lambda x, \lambda y) \to -\infty$ as $\lambda \to 0$, all $(x, y) \in A \backslash \{0\}$. If X is a closed set, then A is also closed, and this restriction is equivalent to assuming that F takes only strictly negative values. If X is not closed, it also requires that the return function not approach zero at the boundary of A. The restriction $-b<0$ is used only in showing that the fixed point of T is unique.

Assumption 2e will be used to show that v^* is continuous. As noted above, since returns may diverge along some feasible paths, u need not be continuous on Π, and Berge's theorem cannot be applied to (2). The role of Assumption 2e is to permit a direct proof of continuity. Assumption 2e insures that for any initial condition x_0 and any sequence $\underline{x} \in \Pi(x_0)$ that is feasible from x_0, if $\hat{x}_0$ is an initial condition close to x_0, then there is a sequence $\hat{\underline{x}} \in \Pi(\hat{x}_0)$ that is feasible from $\hat{x}_0$ with the property that $\hat{\underline{x}}$ is close to $\underline{x}$ at every date (in the sup norm). That is, the growth rate of the state is the same in the tails of the two sequences. Assumption 2e accomplishes this by insuring that there is a sequence $\hat{\underline{x}}$ that is a slightly scaled down version of $\underline{x}$ that is feasible. Continuity of Π in the product topology does not guarantee this, as the following example shows. Let $X = \mathbf{R}^2_+$, and suppose that every feasible path consists of points along a fixed ray and that the growth rate along the ray depends on the angle. Formally, let $\Gamma(x) = \{y\colon y = [1+\cos(x)]x\}$. Then for any two initial conditions x_0 and $\hat{x}_0$ on different rays, no matter how close, and any distance D, there is a date T sufficiently large so that $\|x_t - \hat{x}_t\| > D$, all $t \geqslant T$.

If $X \subseteq \mathbf{R}^{\ell}_+$, then Assumption 2e is implied by free disposal. To see this, choose $x \in X$, $y \in \Gamma(x)$, and $\hat{x}$ close to x. Since Γ is continuous, there exists $\hat{y} \in \Gamma(\hat{x})$ with $\hat{y}$ close to y. Moreover, if $\hat{x}$ is sufficiently close to x, we can choose $\hat{y}$ so that each component of $\hat{y}$ is positive if the corresponding component of y is positive. It follows that $\hat{y} > (1-\varepsilon)y$, for some $\varepsilon > 0$.

Under Assumption 2, we have the following analogue of Theorem 1.

THEOREM 3. *If* $(\theta, X, \Gamma, \beta, F)$ *satisfy Assumption* 2, *then* (a)–(c) *of Theorem* 1 *hold for all* $x_0 \neq 0$.

Proof. Clearly v^* is homogeneous of degree θ, and since $F<0$, v^* is bounded above by zero. Assumption 2c implies that for any $x_0 \in X$, the plan $\xi(x_0) \in \Pi(x_0)$ defined by $\xi_{t+1}(x_0) = g(\xi_t(x_0))$, all t, satisfies $\|\xi_t\| \geqslant \zeta^t \|x_0\|$, all t, so

$$(\|\xi_t\|^\theta + \|\xi_{t+1}\|^\theta) \leqslant \zeta^{\theta t} \|x_0\|^\theta (1+\zeta^\theta), \quad \text{all } t \geqslant 0.$$

Hence

$$\begin{aligned} M(x_0) \equiv u(\xi(x_0)) &= \sum_{t=0}^{\infty} \beta^t F(\xi_t, \xi_{t+1}) \\ &\geqslant -B \|x_0\|^\theta (1+\zeta^\theta) \sum_{t=0}^{\infty} (\beta\zeta^\theta)^t \\ &= -B \|x_0\|^\theta \frac{1+\zeta^\theta}{1-\gamma}, \end{aligned}$$

where $\gamma = \beta\zeta^\theta < 1$ and where B is defined by (7). Hence v^* is bounded below by M, and so is bounded in the norm in (3).

As before, Π: $X \to Z$ is compact valued and continuous in the product topology. In addition, the upper contour sets of u are closed. To see this, suppose that the sequence of paths $\{\underline{x}^k\}_{k=1}^{\infty}$, with $u(\underline{x}^k) \geqslant -a$, all k, converges to the path $\underline{x}^0$. Fatou's Lemma ensures that

$$|u(\underline{x}^0)| = |u(\lim_{k\to\infty} \inf \underline{x}^k)| \leqslant \lim_{k\to\infty} \inf |u(\underline{x}^k)|.$$

Since u takes negative values, this implies that

$$u(\underline{x}^0) \geqslant \lim_{k\to\infty} \sup u(\underline{x}^k) \geqslant -a.$$

Hence the supremum in (2) is attained and the set of maximizers is nonempty and compact valued.

To see that v^* is continuous, let $\{x_0^k\} \to x_0^0 \neq 0$ be a sequence of initial conditions, and for each k let $\underline{x}^k \in \Pi(x_0^k)$ be a path that is optimal from x_0^k. Let $\{\underline{x}^{k_n}\} \to \underline{x}^0$ be a convergent subsequence. Since Π is continuous, $\underline{x}^0 \in \Pi(x_0^0)$. Hence another application of Fatou's lemma shows that

$$v^*(x_0^0) \geqslant u(\underline{x}^0) \geqslant \lim_{n\to\infty} \sup u(\underline{x}^{k_n}) = \lim_{n\to\infty} \sup v^*(x_0^{k_n}).$$

The rest of the argument consists of showing that the complementary inequality also holds.

Let $\underline{x}^* \in \Pi(x_0^0)$ attain the supremum from x_0^0. We will show that for any $\eta > 0$ there exists $\Delta > 0$ such that $\|x_0^0 - \hat{x}_0\| < \Delta$ implies $v(x_0^0) - \eta < v(\hat{x}_0)$. Fix $\eta > 0$, and choose $\varepsilon > 0$ and $\hat{\delta} > 0$ so

$$|F(x_0^0, x_1^*) - F(\hat{x}_0, (1-\varepsilon)x_1^*)| < \frac{\eta}{2}, \quad \text{all } \hat{x}_0 \quad \text{with} \quad \|x_0^0 - \hat{x}_0\| < \hat{\delta},$$

and

$$|1-(1-\varepsilon)^\theta| \sum_{t=1}^{\infty} \beta^t \, |F(x_t^*, x_{t+1}^*)| < \frac{\eta}{2}.$$

Since F is continuous and the series converges, this is possible. Then choose $\delta > 0$ so that Assumption 2e holds for the chosen ε, and define $\Delta \equiv \min\{\delta, \hat{\delta}\}$. Then choose any $\hat{x}_0$ with $\|x_0^0 - \hat{x}_0\| < \Delta$, and define $\hat{x}_1, \hat{x}_2, \ldots$ by

$$\hat{x}_t = (1-\varepsilon)x_t^*, \qquad t = 1, 2, \ldots. \tag{8}$$

Assumption 2e insures that $\hat{x}_1 \in \Gamma(\hat{x}_0)$. Then since Γ is homogeneous of degree one,

$$x_{t+1}^* \in \Gamma(x_t^*) \qquad \text{implies} \quad \hat{x}_{t+1} \in \Gamma(\hat{x}_t), \qquad t = 1, 2, \ldots.$$

Hence $(\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots) = \hat{\underline{x}} \in \Pi(\hat{x}_0)$. Moreover,

$$\begin{aligned}|u(\underline{x}^*) - u(\hat{\underline{x}})| &\leqslant \sum_{t=0}^{\infty} \beta^t \, |F(x_t^*, x_{t+1}^*) - F(\hat{x}_t, \hat{x}_{t+1})| \\ &= |F(x_0^0, x_1^*) - F(\hat{x}_0, (1-\varepsilon)x_1^*)| \\ &\quad + |1-(1-\varepsilon)^\theta| \sum_{t=1}^{\infty} \beta^t \, |F(x_t^*, x_{t+1}^*)| \\ &< \eta.\end{aligned}$$

Hence

$$v^*(\hat{x}_0) \geqslant u(\hat{\underline{x}}) \geqslant u(\underline{x}^*) - \eta = v^*(x_0^0) - \eta.$$

Since $\eta > 0$ was arbitrary, it follows for any sequence of initial conditions $\{x_0^k\} \to x_0^0 \neq 0$,

$$\lim_{n \to \infty} \inf v^*(x_0^{k_n}) \geqslant v^*(x_0^0).$$

Hence v^* is continuous, except at the origin.

Since $v^* \leqslant 0$,

$$\lim_{t\to\infty} \sup \beta^t v^*(x_t) \leqslant 0, \quad \text{all } \underline{x} \in \Pi(x_0), \quad \text{all } x_0 \in X, \tag{9}$$

so claims (b) and (c) follow as before. ▮

Our next task is to study the Bellman equation. As noted above, however, when $\theta < 0$ the operator T need not be a contraction, and Berge's theorem cannot be used to establish that T preserves continuity. Before proving an analogue of Theorem 2, we will discuss each of these problems.

Under Assumption 2 the operator T is not necessarily a contraction, as the following example illustrates. Let $X = \mathbf{R}_+$; let the feasible set be

$$\Gamma(x) = \{y\colon px \leqslant y \leqslant Px\},$$

where $0 \leqslant p < P < 1$; let the return function be

$$F(x, y) = -(x-y)^{-1};$$

and note that $\theta = -1$. For $0 < \beta < P$, this problem satisfies all the conditions of Assumption 2: choose $g(x) = Px$, $\zeta = \beta P^{-1}$, and $0 < b < (1-p)^{-1}$, and let $B = (1-P)^{-1}(1+P^{-1})^{-1}$.

Define T by (6), and note that any function $f \in H^-(X, \theta)$ has the form $f(x) = -ax^{-1}$, where $a > 0$. For $f(x) = -ax^{-1}$ the optimal policy in (6) is

$$y = \begin{cases} xp, & \text{if } \ a < \underline{a}, \\ x\sqrt{\beta a}/(1+\sqrt{\beta a}), & \text{if } \ \underline{a} \leqslant a \leqslant \bar{a}, \\ xP, & \text{if } \ a > \bar{a}, \end{cases}$$

where $\underline{a} = (1/\beta)(p/(1-p))^2$ and $\bar{a} = (1/\beta)(P/(1-P))^2$. If a lies in the intermediate range, then

$$Tf(x) = -h(a, \beta)x^{-1},$$

where

$$h(a, \beta) \equiv (1 + 2\sqrt{\beta a} + \beta a).$$

Let $f_i(x) = -a_i x^{-1}$, $i = 1, 2$ with $\underline{a} = a_1 < a_2 \leqslant \bar{a}$. Then

$$\begin{aligned} \frac{\|Tf_2 - Tf_1\|}{\|f_2 - f_1\|} &= \frac{|h(a_2, \beta) - h(a_1, \beta)|}{|a_2 - a_1|} \\ &= 2\sqrt{\beta}\left(\frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1}\right) + \beta, \end{aligned}$$

and

$$\lim_{a_2 \searrow a_1} 2\sqrt{\beta}\left(\frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}\right)+\beta=\frac{\sqrt{\beta}}{\sqrt{a_1}}+\beta=\frac{\sqrt{\beta}}{\sqrt{\underline{a}}}+\beta=\frac{\beta}{p}.$$

If $p<\beta$, this expression exceeds unity and T is not a contraction.

Moreover, under Assumption 2 the term in brackets on the right side of (6) can take the value $-\infty$. For example, since $f(0)=-\infty$ for any relevant f, this happens whenever $0 \in \Gamma(x)$. Hence Berge's theorem cannot be used to establish that the operator T maps continuous functions to continuous functions. Instead we will use Lemma 2, which is a modification of Berge's theorem. The key assumptions are that the return function is continuous everywhere that it is finite valued, and that there exists a continuous feasible policy that delivers a finite return. The proof involves defining a modified correspondence, constructed by restricting attention to choices for which returns are no less than under the postulated feasible policy, and showing that this modified correspondence is everywhere u.h.c. and satisfies a condition like l.h.c at points in the interior of its graph. The rest of the proof is then similar to the proof of Berge's theorem.[1]

LEMMA 2. *Let* (X, ρ) *and* (Y, μ) *be metric spaces; let* $Q: X \rightarrow Y$ *be a (nonempty) compact-valued and continuous correspondence; let* A *be the graph of* Q; *and let* $\phi: A \rightarrow R \cup \{-\infty\}$. *Assume that* ϕ *is continuous at every point in* A *where* $\phi(x, y)$ *is finite, and that there is a continuous selection* q *from* Q *such that*

$$\phi(x, q(x)) > -\infty, \qquad \text{all} \quad x \in X.$$

Then the function $h: X \rightarrow \boldsymbol{R}$ *defined by*

$$h(x)=\sup_{y \in Q(x)} \phi(x, y) \tag{10}$$

is continuous, and the correspondence $G: X \rightarrow Y$ *defined by*

$$G(x)=\{y \in Q(x): \phi(x, y)=h(x)\},$$

is nonempty, compact valued, and u.h.c.

With this result in hand, an analogue of Theorem 2 still holds.[2] Lemma 2 will be used to show that T preserves continuity; an alternative argument

[1] A complete proof is available upon request from the authors.

[2] Unfortunately, Lemma 2 does not appear to be useful in establishing that v^* is continuous. The problem is that it applies to $v^*(x_0) \equiv \sup_{\underline{x} \in \Pi(x_0)} u(\underline{x})$ only if we can establish that u is continuous everywhere that it is finite valued. But as noted above, the Lebesgue Dominated Convergence Theorem does not apply here, and we could not find an alternative argument showing that u is continuous.

will establish that v^* is the unique solution to the Bellman equation; and monotonicity of T will be used to show that the sequence $\{T^n f\}$ converges to v^* for a certain class of initial functions f.

THEOREM 4. *Let $(\theta, X, \Gamma, \beta, F)$ satisfy Assumption 2, and define the operator T on $H^-(X, \theta)$ by (6). Then*

(a) *$T: H^-(X, \theta) \to H^-(X, \theta)$, v^* is the unique fixed point of T in $H^-(X, \theta)$, and $T^n f \to v^*$ in the following cases:*

(i) *if f is the zero function, $f(x) \equiv 0$, or*

(ii) *if $f = h \in H^-(X, \theta)$ is the total discounted return from following a (stationary) feasible policy, i.e., a policy of the form $g_h(x) \in \Gamma(x)$, all x, where g_h is continuous.*

(b) *Part (b) of Theorem 2 holds.*

Proof. Fix $f \in H^-(X, \theta)$. Clearly Tf is homogeneous. To see that Tf is continuous, choose g satisfying Assumption 2c, and apply Lemma 2 with (Γ, F, g) for (Q, ϕ, q). For boundedness, note that

$$\begin{aligned}(Tf)(x) &= \sup_{y \in \Gamma(x)} [F(x, y) + \beta f(y)] \\ &\geqslant F(x, g(x)) + \beta f(g(x)) \\ &\geqslant -[B(1 + \zeta^\theta) + \beta \zeta^\theta \, \|f\|] \, \|x\|^\theta,\end{aligned}$$

where B is defined in (7). Hence Tf is bounded, and T maps $H^-(X, \theta)$ into itself.

Since v^* satisfies (1), clearly it is a fixed point of T. To establish uniqueness, suppose $\hat{v}, \tilde{v} \in H^-(X, \theta)$ are fixed points of T, and $\hat{v}(x_0) \neq \tilde{v}(x_0)$ for some $x_0 \in X$. Without loss of generality, label the functions so that $\tilde{v}(x_0) - \hat{v}(x_0) = \varepsilon > 0$, and let $\{\tilde{x}_t\}$ be a sequence that attains $\tilde{v}(x_0)$. That is,

$$\tilde{v}(x_0) = \sum_{t=0}^{n-1} \beta^t F(\tilde{x}_t, \tilde{x}_{t+1}) + \beta^n \tilde{v}(\tilde{x}_n), \qquad \text{all } n. \tag{12}$$

Assumption 2d implies

$$F(x, y) \leqslant -b \, \|x\|^\theta < 0, \qquad \text{all } (x, y) \in A,$$

so it follows from the negativity of F and $\tilde{v}$ that

$$|\tilde{v}(x_0)| = \sum_{t=0}^{n-1} \beta^t |F(\tilde{x}_t, \tilde{x}_{t+1})| + \beta^n |\tilde{v}(\tilde{x}_n)| \geqslant b\left[\sum_{t=0}^{n-1} \beta^t \|\tilde{x}_t\|^\theta\right], \quad \text{all } n.$$

Since $|\tilde{v}(x_0)|$ is finite, it follows that

$$\beta^t \|\tilde{x}_t\|^\theta \to 0 \qquad \text{as} \quad t \to \infty. \tag{13}$$

Also, since $\tilde{v} \leqslant 0$, and $\hat{v}$ is homogeneous of degree θ,

$$\tilde{v}(x) - \hat{v}(x) \leqslant -\hat{v}(x) = \|x\|^\theta \left|\hat{v}\left(\frac{x}{\|x\|}\right)\right| \leqslant \|x\|^\theta \|\hat{v}\|, \quad \text{all } x. \tag{14}$$

Finally, since $\hat{v}$ satisfies the Bellman equation,

$$\hat{v}(x_0) = \sup_{\{x_t\}} \left[\sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n \hat{v}(x_n)\right], \quad \text{all } n, \tag{15}$$

where the supremum is over feasible sequences. Hence for all n,

$$\begin{aligned}
\varepsilon &= \tilde{v}(x_0) - \hat{v}(x_0) \\
&= \left[\sum_{t=0}^{n-1} \beta^t F(\tilde{x}_t, \tilde{x}_{t+1}) + \beta^n \tilde{v}(\tilde{x}_n)\right] - \sup_{\{x_t\}} \left[\sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n \hat{v}(x_n)\right] \\
&\leqslant \beta^n [\tilde{v}(\tilde{x}_n) - \hat{v}(\tilde{x}_n)] \\
&\leqslant \beta^n \|\tilde{x}_n\|^\theta \|\hat{v}\|,
\end{aligned}$$

where the second line uses (12) and (15), the third uses the fact that $\{\tilde{x}_t\}$ is feasible from x_0, and the last uses (14). But $\|\hat{v}\|$ is fixed, so it follows from (13) that the last line converges to zero as $n \to \infty$, a contradiction. Hence T cannot have more than one fixed point.

Clearly T is a monotone operator. Let f be the zero function, $f(x) \equiv 0$. Then $Tf \leqslant f$ (from the fact that $F \leqslant 0$), $v^* \leqslant Tv^* \leqslant Tf$ (from monotonicity), and so by induction $v^* \leqslant T^{n+1} f \leqslant T^n f$, all n. The sequence $\{T^n f\}$ is monotone decreasing and bounded below, so the limit exists. Define $f^\infty \equiv \lim_{n \to \infty} T^n f$. The arguments showing that $f^\infty \in H^-(X, \theta)$, and that f^∞ is a fixed point of T are standard. Hence $f^\infty = v^*$.

Similarly let $g_h(x)$ be a continuous feasible policy function that attains the return $h(x)$, all x. Then $h \leqslant Th$, since maximizing gives a return at least as great as the feasible action g_h; $Th \leqslant Tv^* = v^*$, from monotonicity; and so by induction $T^n h \leqslant T^{n+1} h \leqslant v^*$, all n. That is, the sequence $\{T^n h\}$ is increasing and is bounded above by v^*, so the limit, call it h^∞, exists. As before, $h^\infty \in H^-(X, \theta)$ is a fixed point of T, so $h^\infty = v^*$.

The proof of (b) is the same as for Theorem 2. ∎

4. LOGARITHMIC FUNCTIONS: $\theta = 0$

For the logarithmic case the main issue is selecting a suitable space of functions. With this done, the argument involves combining elements from both previous cases to establish upper and lower bounds and continuity. As in the case $\theta < 0$, u need not be continuous (because it may diverge to minus infinity), so Berge's theorem cannot be applied to (2). In this case the operator T is a contraction, however, so we can establish the continuity of v^* indirectly using the argument sketched at the end of Section 2. That is, we show that T preserves continuity. Hence, T is a contraction on the space of continuous functions, a complete space, so v^* is continuous. To show that T preserves continuity we use Lemma 2, instead of Berge's theorem, since F and f are unbounded below.

Let $X \subset \mathbf{R}^\ell$ be a cone, and for any $\beta \in (0, 1)$ consider the space of functions f: $X \to \mathbf{R}$ that satisfy

$$f(x) = f\left(\frac{x}{\|x\|}\right) + \frac{\ln \|x\|}{1-\beta}, \quad \text{all } x \in X, \quad x \neq 0. \tag{16}$$

For any $a \in \mathbf{R}$, define the constant function $a(x) = a + \ln \|x\|/(1-\beta)$. In particular, the zero function is $o(x) = \ln \|x\|/(1-\beta)$. Define addition by

$$(f+g)(x) = f\left(\frac{x}{\|x\|}\right) + g\left(\frac{x}{\|x\|}\right) + \frac{\ln \|x\|}{1-\beta},$$

and scalar multiplication by

$$\alpha f(x) = \alpha f\left(\frac{x}{\|x\|}\right) + \frac{\ln \|x\|}{1-\beta}.$$

Clearly

$$\|f\| = \sup_{\|x\| = 1,\, x \in X} |f(x)| \tag{17}$$

defines a norm on this space. Let $B_\beta(X, 0)$ denote the space of functions satisfying (16) that are bounded in the norm in (17), and let $H_\beta(X, 0) \subset B_\beta(X, 0)$ denote the subspace containing the continuous functions. With the norm in (17), clearly $B_\beta(X, 0)$ is complete, and since $H_\beta(X, 0)$ is a closed subset, it is also complete. As before, the key idea is that functions in $B_\beta(X, 0)$ and $H_\beta(X, 0)$ are characterized by their behavior on the intersection of X with the unit circle. Also note that for any $f \in B_\beta(X, 0)$,

$$f(x) \leqslant \|f\| + \frac{\ln \|x\|}{1-\beta}. \tag{18}$$

When the return function is logarithmic it is unbounded both above and below. Hence the growth rate of $\|x_t\|$ must be bounded above for every feasible sequence, and for each $x_0 \in X$, there must be at least one feasible sequence along which the growth rate of $\|x_t\|$ is bounded below. In addition, the ratio of the return function to the norm of its arguments must be bounded from above everywhere and bounded from below along the feasible path where the growth rate is bounded below. Thus, the following assumption combines the restrictions used in the previous cases.

Assumption 3. a. $X \subseteq \mathbf{R}^\ell$ is a cone;

b. the correspondence $\Gamma: X \to X$ is nonempty, compact-valued, and continuous, and the graph of Γ, call it A, is a cone;

c. $\beta \in (0, 1)$, and there exists a continuous and homogeneous of degree one selection g from Γ and numbers $0 < \zeta < \alpha < +\infty$ such that

$$\|g(x)\| \geqslant \zeta \|x\|, \quad \text{all } x \in X,$$
$$\|y\| \leqslant \alpha \|x\|, \quad \text{all } y \in \Gamma(x), \quad \text{all } x \in X;$$

d. $F: A \backslash \{0\} \to \mathbf{R}$ has the property that $F(x, y) = \ln \phi(x, y)$, where $\phi: A \backslash \{0\} \to \mathbf{R}_+$ is continuous and homogeneous of degree one; $F(0, 0) = -\infty$; and there exist $0 < b < B < +\infty$ such that

$$b \|x\| \leqslant \phi(x, g(x)), \quad \text{all } x \in X,$$
$$\phi(x, y) \leqslant B(\|x\| + \|y\|), \quad \text{all } y \in \Gamma(x), \quad \text{all } x \in X.$$

Part d is equivalent to assuming that for $\|x\| = 1$, $\phi(x, y)$ is bounded above for all $y \in \Gamma(x)$ and bounded away from zero for $y = g(x)$.

THEOREM 5. *If* (X, Γ, β, F) *satisfy Assumption* 3, *then* $v^* \in B_\beta(X, 0)$ *and* (b) *and* (c) *of Theorem* 1 *hold.*

Proof. Assumption 3c implies that $\|x_t\| \leqslant \alpha^t \|x_0\|$, for all t, all $\underline{x} \in \Pi(x_0)$, and all $x_0 \in X$, so

$$\begin{aligned} \ln(\|x_t\| + \|x_{t+1}\|) &\leqslant \ln(\alpha^t(1+\alpha) \|x_0\|) \\ &= t \ln \alpha + \ln \|x_0\| + \ln(1+\alpha), \quad \text{all } t. \end{aligned} \tag{19}$$

It then follows from Assumption 3d that

$$\beta^t F(x_t, x_{t+1}) \leqslant \beta^t \ln(B(\|x_t\| + \|x_{t+1}\|))$$
$$\leqslant \beta^t [t \ln \alpha + \ln \|x_0\| + \ln(1+\alpha) + \ln(B)], \quad \text{all } t.$$

Hence total returns are bounded above along every feasible path. As before Assumptions 3c and 3d imply total returns are bounded below along the feasible path $\xi(x_0) \in \Pi(x_0)$ defined iteratively by g. Hence v^* is bounded in the norm in (17). And since Π is homogeneous of degree one and $u(\underline{x})$ is homogeneous in the sense that $u(\lambda \underline{x}) = u(\underline{x}) + \ln(\lambda)/(1-\beta)$, it follows that v^* satisfies (16).

Finally, (18) and (19) together imply that (9) holds, so claims (b) and (c) follow as before. ∎

In this case a contraction argument applies and the conclusions of Theorem 2 go through without change. The next lemma provides the analogue of Boyd's [4] result that is suitable for the logarithmic case. As before the lemma holds for $H_\beta(X, 0)$ but also more generally. In the application here $\gamma = \beta$, but the lemma does not require this.

LEMMA 3. *Let $X \subseteq \mathbf{R}^\ell$ be a cone; let $\beta \in (0, 1)$; let $J_\beta(X, 0)$ be a space of functions that satisfy* (16) *and are bounded in the norm in* (17). *Let T: $J_\beta(X, 0) \to J_\beta(X, 0)$ be an operator satisfying* (a) *and* (b) *of Lemma* 1. *Then T is a contraction of modulus γ.*

THEOREM 6. *Let (X, Γ, β, F) satisfy Assumption* 3, *define $H_\beta(X, 0)$ as above, and define the operator T on $H_\beta(X, 0)$ by* (6). *Then*

(a) *T: $H_\beta(X, 0) \to H_\beta(X, 0)$, T is a contraction of modulus β, and the supremum function v^* for* (2) *is the unique fixed point of T in $H_\beta(X, 0)$.*

(b) *Part* (b) *of Theorem* 2 *holds.*

Proof. The argument that T: $B_\beta(X, 0) \to B_\beta(X, 0)$ is standard, and it follows immediately from Lemma 3 that T is a contraction. Hence T has a unique fixed point in $B_\beta(X, 0)$, call it $\hat{v}$. Since v^* is clearly a fixed point of T, it follows immediately that $\hat{v} = v^*$. The argument that T preserves continuity, which uses Lemma 2 and the function g in Assumption 3c, is the same as in the proof of Theorem 4. Hence T: $H_\beta(X, 0) \to H_\beta(X, 0)$. Since $H_\beta(X, 0)$ is a closed subset of $B_\beta(X, 0)$, it follows that $v^* \in H_\beta(X, 0)$. The rest of the argument is as before. ∎

5. EXTENSIONS

Arguments analogous to those in [12, Section 4.3] can be used to establish (weak or strict) concavity of the value function. Several additional assumptions are needed: that X is convex, that $\Gamma(x)$ is convex-valued, for all $x \in X$, and that F or ϕ is (weakly or strictly) quasi-concave. The arguments above can then be applied to the closed subset of H, H^- or H_β consisting of functions that are weakly quasi-concave.

The arguments above can be extended in a straightforward way to the case where only some of the state variables display sustained growth. The technology adoption model analyzed in Parente [10] has this form. Suppose that the state has two components, $x = (x_1, x_2)$, and only x_1 displays sustained growth. The state space is $X \subset X_1 \times X_2$, where X_1 is a cone and X_2 is a compact set. The state space is no longer a cone, but assume that for each x_2,

$$(x_1, x_2) \in X \Rightarrow (\lambda x_1, x_2) \in X, \quad \text{all } \lambda > 0.$$

For $\theta \leqslant 1$ and $\theta \neq 0$, let $H(X, \theta)$ be the linear space of functions $f: X \to \mathbf{R}$ that are continuous, homogeneous of degree θ in x_1, and bounded in the norm

$$\|f\| = \sup_{\|x_1\| = 1,\, (x_1, x_2) \in X} |f(x_1, x_2)|.$$

For the logarithmic case, let $H_\beta(X, 0)$ be the space of continuous functions that satisfy

$$f(x_1, x_2) = f\left(\frac{x_1}{\|x_1\|}, x_2\right) + \frac{\ln \|x_1\|}{1-\beta}.$$

Assumptions 1–3 must be modified in the obvious way. For example, if $\theta \in (0, 1]$, assume that for each $(x_1, x_2) \in X$,

$$(y_1, y_2) \in \Gamma(x_1, x_2) \Rightarrow (\lambda y_1, y_2) \in \Gamma(\lambda x_1, x_2), \quad \text{all } \lambda > 0;$$

that there exists $\alpha > 0$, with $\gamma \equiv \beta \alpha^\theta < 1$, such that

$$\|y_1\| \leqslant \alpha \|x_1\|, \quad \text{all } (x_1, x_2, y_1, y_2) \in A;$$

and that the return function $F(x_1, x_2, y_1, y_2)$ is homogeneous of degree θ in (x_1, y_1), and for some $0 < B < \infty$,

$$F(x_1, x_2, y_1, y_2) \leqslant B(\|x_1\| + \|y_1\|)^\theta, \quad \text{all } (x_1, x_2, y_1, y_2) \in A.$$

To simplify the exposition, the deterministic case was discussed here. The same arguments apply to stochastic models, however. Suppose the state is $s=(x, z)$, where x is a vector of endogenous state variables and z a vector of exogenous shocks, so the state space is $S=X\times Z$, where X is a cone and Z is a compact set. For $\theta\leqslant 1$ and $\theta\neq 0$, let $H(S, \theta)$ be the linear space of functions $f\colon S\to\mathbf{R}$ that are continuous, homogeneous of degree θ in x, and bounded in the norm

$$\|f\| = \sup_{\|x\|=1,\ (x, z)\in S} |f(x, z)|.$$

In this case the feasible set must satisfy

$$y\in\Gamma(x, z)\Rightarrow \lambda y\in\Gamma(\lambda x, z), \quad \text{all } \lambda>0, \quad \text{all } x\in X, \quad \text{all } z\in Z;$$

the bounds on the growth rate of $\|x_t\|$ and on the ratio of F or ϕ to the norm of its arguments must hold contingent on every realization of the exogenous stochastic shock; and the return function $F(x, y, z)$ must be homogeneous of degree θ in the pair (x, y).

Using the results above, it is also easy to establish the necessity of the transversality conditions for homogeneous problems. In general, this is difficult to do for problems that are unbounded below, and problems that are homogeneous of degree $\theta\leqslant 0$ are one such class. We sketch the proof for the case where F is differentiable.

Let Assumption 1, 2, or 3 hold (depending on the value of θ), and assume that X is convex, that $\Gamma(x)$ is convex-valued, for all $x\in X$, and that F is quasi-concave. In addition, assume that F is continuously differentiable, let $F_x(x, y)$ denote the derivative with respect to its first (vector) argument, and assume that $F_x(x, y)\cdot x\geqslant 0$, all $(x, y)\in A$. Consider the problem in (2), and let v^* denote the supremum function. We say that a sequence $\{x_t\}_{t=1}^{\infty}$ satisfies the transversality condition if

$$\lim_{t\to\infty} \beta^t F_x(x_t, x_{t+1})\cdot x_t=0. \tag{20}$$

By Theorem 4.11 in [12], if $\{x_t\}$ is an interior solution to (2), then $v^{*\prime}(x_t)=F_x(x_t, x_{t+1})$, all t. Euler's theorem for homogeneous functions, applied to v^*, says that

$$v^{*\prime}(x)\cdot x=\begin{cases}(\theta-1)\, v^*(x), & \text{for } \theta\neq 0,\\ 1/(1-\beta), & \text{for } \theta=0.\end{cases}$$

Hence

$$0 \leqslant \beta^t F_x(x_t, x_{t+1}) \cdot x_t = \beta^t v^{*\prime}(x_t) \cdot x_t$$
$$= \begin{cases} \beta^t(\theta - 1)\, v^*(x_t), & \text{for } \theta \neq 0, \\ \beta^t/(1-\beta), & \text{for } \theta = 0. \end{cases} \tag{21}$$

For $\theta > 0$, it was proved in Theorem 1 that

$$\lim_{t \to \infty} \beta^t v^*(x_t) = 0;$$

and for $\theta < 0$, it was proved in Theorem 4. For $\theta = 0$, it follows immediately from Assumption 3c that

$$\lim_{t \to \infty} \frac{\beta^t}{1-\beta} = 0.$$

In all three cases, then, it follows from (21) that (20) holds. If F is not differentiable, the necessity of the transversality condition (appropriately restated) can still be proved, but the notation is more cumbersome.

Dolmas [6] shows that return functions that are recursive and homogeneous are also consistent with balanced growth, even if they are not additively separable over time. As shown in [1, Ch. 3], the arguments above can also be extended to this more general class.

The results of Boldrin and Montrucchio in [3] can also be extended to the class of homogeneous dynamic programming problems. Although their theorems, which are for bounded problems in a compact state space, do not apply directly to the case of homogeneous (and hence unbounded) problems, the results and proofs are very similar, and can be found in [1, Ch. 1]. The main result is that for any $\theta \neq 0$, $\theta < 1$, and any function g: $X \to X$ that is homogeneous of degree one, is C^2, and satisfies a certain a growth condition, there exists a nontrivial dynamic programming problem, homogeneous of degree θ, for which g is the optimal policy function. That is, there is a correspondence Γ that is homogeneous of degree one, a return function F defined on the graph of Γ that is homogeneous of degree θ, strictly concave, increasing in the vector of current states, and decreasing in the vector of next period states, and a discount factor $\beta > 0$, such that the problem defined by (Γ, F, β) has the given function g as its optimal policy rule.[3]

The growth condition on g rules out optimal paths along which the rate of change in the norm of the state vector diverges, or the relative magnitude

[3] The problem can be chosen to be nontrivial in the sense that $g(x) \in \text{int } \Gamma(x)$, all $x \in X$, so that $g(x)$ is not the only possible choice of y given x.

of any component of the state vector diverges. Formally the restriction on g is that there exist d, D, e, E such that

$$0 < d \leqslant \frac{\| g(x) \|}{\| x \|} \leqslant D < \infty, \quad \text{all } x \in X,$$

and

$$0 < e \leqslant \frac{g_i(x)}{\| g(x) \|} \leqslant E < \infty, \qquad i = 1, ..., \ell, \quad \text{all } x \in X,$$

where ℓ is the number of components of the state vector.

Homogeneous models have the feature, attractive for computational purposes, that the dimensionality of the problem is essentially reduced by one. Since both the value and policy functions are homogeneous, it suffices to compute their values on the unit circle. Thus, for a problem with state space $X \subset \mathbf{R}^{\ell}$, it is enough to compute the value and policy function on a manifold of dimension $\ell - 1$. Hence the "curse of dimensionality" operates a little more slowly.

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